Formalization of matching numbers with `finmap` and `mathcomp-classical`

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Rocqshop 2025, Reykjavik, September 27, 2025

We will present our formalization of graph theory in Rocq, which is intended for expressing and proving integer invariants of graphs. Namely, we have formalized variations of *matching* subsets of edges of a graph, showing that their maximum or minimum sizes satisfy various inequalities. Such inequalities were described without formalization in the context of commutative algebra [2–5]. Our motivation lies in finding more general or precise results supported by a proof checker.

A notable related work is the `graph-theory` library by Doczkal et al. [1], which is based on the SsReflect library of Rocq, and contains the formalization of Menger's theorem as well as Robertson and Seymour's graph minor theorem. While our work is also using Rocq and SsReflect, it differs technically in two ways: 1. our graph definition is based on a boundary function that maps an edge to its boundary points (source and target), unlike the one based on a boolean relation in `graph-theory`; 2. we use more components from MathCompfamily of libraries, especially the `mathcomp-finmap` and `mathcomp-classical` packages, and leverage classical reasoning to deal with sets of edges or vertices. Whether these technicalities can lead to differences in flexibility or ease of reasoning would be an interesting question, whose answer we want to seek by presenting our work.

We show some of our code below. One can see that we are using finite sets from the `mathcomp-finmap` package both for expressing the results of the boundary function and the definitions of matching sets. We plan to migrate the latter into classical sets from the `mathcomp-classical` package for possibly simpler definitions and proofs. We will include a comparison between these approaches in our presentation.

A graph consists of: V : type of vertices, E : type of edges, $d : e \to 2^V$, and an axiom $|d(x)| = 2$.

```
Record t := mk {              (* Notation llugraph *)
  V : finType;                (* Notation `V *)
  E : finType;                (* Notation `E *)
  boundary : E -> {fset V}; (* Notation `d *)
  _ : forall e : E, #|` boundary e | = 2;
}.
```

For a graph (V, E, d) and $S \subset E$, S is a

matching if no two edges share a vertex, and an

induced matching if furthermore no two edges are connected by an edge

```
Definition is_matching S :=
 [∀ e in S, [∀ f in S, (e != f) ==> [`d(e) ⊥ `d(f)]]].
Definition is_induced_matching S :=
 [∀ e in S, [∀ f in S,
   (e != f) ==> [∀ g, [`d(e) ⊥ `d(g)] || [`d(f) ⊥ `d(g)]]]].
(* [ X ⊥ Y ] = X and Y are disjoint *)
```

These sets are accompanied by integer invariants:

matching number is the maximum size (cardinality) of a matching

induced matching number is the maximum size of an induced matching

minimal matching number is the minimum size of a maximal matching

```
Definition nmatch := \max_(S∈matching) #|` S |.
Definition nindmatch := \max_(S∈induced matching) #|` S |.
Definition is_maximal_matching :=
  (S ∈ matching) && [∀ T : {fset `E}, (S ⊂ T) ==> (T ∉ matching)].
Definition nminmatch := \big[min/nmatch]_(S∈maximal matching) #|` S |.
```

One initial goal of our formalization was the following inequality, which has significance in the context of commutative algebra [2, Proposition 2.1]:

$$\texttt{nindmatch} \leq \texttt{nminmatch} \leq \texttt{nmatch} \leq 2\texttt{nminmatch}.$$

Our recent development includes formalization of notions such as complete graphs, independence numbers, neighborhoods, etc., which can also be mentioned to illustrate our uses of MathComp for reasoning on sets.

References

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[2] T. Hibi, A. Higashitani, K. Kimura, and A. Tsuchiya. Dominating induced matchings of finite graphs and regularity of edge ideals. *Journal of Algebraic Combinatorics*, 43(1):173–198, 2016.

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