Generating Higher Identity Proofs in Homotopy Type Theory

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Types in Martin-Löf type theory with intentional equality [18] are weak ω-groupoids. This is one of the foundational observation of homotopy type theory [19] (HoTT). Finster and Mimram [12] have introduced the dependent type theory CaTT, which encodes the structure of weak ω-categories, a particular kind of ω-groupoid with additional directedness conditions. CaTT and HoTT are both dependent type theories, the former is a theory of ω-categories while in the latter, every type is an ω-groupoid, so in particular, an ω-category. We establishe a formal connection between the two by defining an interpretation of CaTT terms into HoTT, making explicit that identity types implement the structure of weak ω-categories. We view higher cells in CaTT as higher identity proofs in HoTT, and prove the correctness of this translation. This means that the translation of a well-formed term in context in CaTT is a well-formed term in HoTT, whose type we characterise as the translation of the type of the original term in CaTT. In the terminology of Boulier, Pedrot and Tabareau [8], HoTT is a syntactic model of CaTT. The translation is defined by induction on the syntax of CaTT, and the main difficulty is to translate each of the operation making the ω-category structure into successive applications of the J rule in HoTT. We also present an implementation of this translation that generates terms in Rocq, in the form of a Rocq plugin. In combination with mechanised reasoning available in CaTT to reduce the proof effort required to generate certain complex terms in HoTT.

The theory CaTT and proof mechanisation. The definition the theory CaTT by Finster and Mimram [12] was inspired by both Brunerie's formulation of weak ω-groupoids in dependent type theory, and Maltsiniotis' definition of weak ω-categories from weak ω-groupoids. Several works have been conducted around this theory. It was generalised by Dean et al. [11] to give an inductive definition of computads for weak ω-categories, it was proven by Benjamin, Finster and Mimram [5] that the models of CaTT are precisely Grothendieck-Maltsiniotis ω-categories, and several meta-operations for CaTT have been proposed, such as the suspension and opposites [6], computation of inverses and invertibility witnesses [7], functorialisation [4], allowing for mechanised reasoning in the proof

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assistant CaTT. Our proposed translation allows to leverage this existing body of work to access proof mechanisation on identity types in HoTT.

Main results. We define a translation $[\![\_]\!]$ which produces a judgment in HoTT from one in CaTT, and show the following correctness result:

Theorem. *Given a derivable term $\Gamma \vdash t : A$ in CaTT, the following judgement is derivable in HoTT:*

$$\emptyset \vdash [\![\Gamma \vdash t : A]\!] : [\![\Gamma \vdash A \ \mathsf{type}]\!].$$

We also provide a Rocq plugin in order that implements this translation, and evaluate the size of the generated in comparison with terms proving the same results in the HoTT library of Rocq. In absence of better metric, we measure the size by shear number of characters needed to print out the proof term, deactivating all syntactic sugar and custom notations.

	HoTT library	generated from CaTT
proof term	110,940	26,427
source file	N/A	786

Related works. Weak ω-groupoids were first defined by Grothendieck [13] with the intent of modelling spaces up to homotopy. The connection between types and groupoids was first discovered by Hofmann and Streicher [14]. Lumsdaine [16], Van den Berg and Garner[20] and Altenkirch and Rypacek [1] then promoted this to ω-groupoids, showing that types with their iterated identity types are endowed with the structure of weak ω-groupoids. While mathematically weak ω-groupoids are described as a collection of cells in every dimension that allow for partial composition operations satisfying associativity, unitality and exchange laws up to higher cells, in Martin-Löf type theory, the entire structure simply emerges from the J rule. With a careful examination of how the J rule yields a weak ω-groupoid structure, Brunerie [10] proposed a definition of ω-groupoids formulated as a dependent type theory, as well as an interpretation of this theory in HoTT. We extend his work to give an interpretation of a theory for ω-categories, which is implemented in a proof assistant and gives access to several mechanised reasoning procedures.

Weak ω-categories were first defined by Batanin [3]. The definition was then elaborated upon by Leinster [15]. More recently, Maltsiniotis [17] proposed an alternative definition, which is based on Grothendieck's definition of groupoids and modifies it by enforcing a privileged direction. This definition has been proven equivalent to that of Batanin and Leinster by Ara [2] and Bourke [9].

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