Can States Be Decidable in Inquisitive Mechanizations?

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1 Introduction

Inquisitive logic [4, 2] provides a framework for studying both declarative and interrogative sentences in one setting using *inquisitive disjunction* $\vee\!\!\!\vee$ and the *inquisitive existential quantifier* $\exists\!\!\!\exists$. Borrowing our examples from Litak and Sano [7] and asuming $S(x, y)$ is the predicate "x sings for y", e denotes Eric Clapton and g denotes Gottlob Frege, we write "Does Eric Clapton sing for Gottlob Frege?" as

$$?S(e, g) := S(e, g) \vee\!\!\!\vee \neg S(e, g).$$

The question "Who is some person that Clapton sings for?" is rendered as $\exists\!\!\!\exists\, x.S(e, x)$, while "Which are the people Clapton sings for?" as $\forall x.?S(e, x)$, and "Is Clapton singing for everybody?" as $?\forall x.S(e, x)$. In its *support semantics*, inquisitive formulae are interpreted by *information states* which are **sets** of classical first-order models (over a fixed domain). These models themselves are considered to be *possible worlds*. Note that one can restrict information states to be finite, which leads to *bounded* inquisitive logic; further restriction of semantics to at most n possible worlds (note no restriction on the cardinality of the domain of individuals!) is called *n-boundedness*.

When mechanizing calculi with such semantics in Rocq, the representation of states is crucial. Using our Autosubst-based [9] formalization [8] of a labelled sequent calculus proposed by Litak and Sano [7] as a case study[1], we discuss whether one can/should use boolean predicates of type `World` $\to$ `bool` (decidable by definition), or rather arbitrary functions of type `World` $\to$ `Prop`.

2 Preliminaries

Definition 1 ( ). The set of *formulae* $\mathcal{F}$ is defined via the BNF

$$\varphi ::= P(\overline{x}) \mid \bot \mid \varphi \to \varphi \mid \varphi \wedge \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid \forall x.\varphi \mid \exists\!\!\!\exists\, x.\varphi$$

where $\overline{x} \in \mathrm{Var}^n$ is a tuple of variables and P is a n-ary symbol from the signature Pred.

Remark 2. We use *arity types* to represent arities of predicate symbols (and function symbols; cf. Footnote 2).

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[1] Wherever applicable, this abstract provides clickable links to our formalization [8] indicated by the Rocq logo .

$$\frac{\alpha \to \varphi \vee\!\!\!\vee \psi}{(\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi)}$$

$$\frac{\alpha \to \exists\!\!\!\exists\, x.\,\psi \qquad x \notin \mathrm{FV}(\alpha)}{\exists\!\!\!\exists\, x.\,\alpha \to \varphi}$$

Figure 1: Split rules

Thus, predicate formulae are implemented via dependent types using argument functions. This motivates a setoid based development in order to provide a suitable equality relation `form_eq` . Consequently, we avoid functional extensionality as additional axiom. Unfortunately, Autosubst [9] uses this axiom internally, forcing us to reprove some theorems regarding substitions, e.g. `hsubst_comp'` .

Definition 3 ( ). A *model* is a tuple

$$\mathfrak{M} := (\mathrm{W}_{\mathfrak{M}}, \mathrm{D}_{\mathfrak{M}}, \mathfrak{I}_{\mathfrak{M}}),$$

where $\mathrm{W}_{\mathfrak{M}}$ is a set of *(possible) worlds*, $\mathrm{D}_{\mathfrak{M}}$ is a non-empty *domain of individuals* and $\mathfrak{I}_{\mathfrak{M}} : \mathsf{Pred} \times \mathrm{W}_{\mathfrak{M}} \to \mathrm{D}_{\mathfrak{M}}^{<\omega}$ sends each n-ary P to $\mathfrak{I}_{\mathfrak{M}}(P, w) \in \mathcal{P}(\mathrm{D}_{\mathfrak{M}}^n)$.

Definition 4 ( ). Given a model $\mathfrak{M}$, any $s \subseteq \mathrm{W}_{\mathfrak{M}}$ is an *information state*. Functions $\eta : \mathrm{Var} \to \mathrm{D}_{\mathfrak{M}}$ are called *variable assignments*. The *support relation* is defined as follows:

$$\begin{aligned}
\mathfrak{M}, s \Vdash_\eta P(\overline{x}) &\text{ if } \overline{\eta(x)} \in \mathfrak{I}_{\mathfrak{M}}(P, w) \text{ for every } w \in s \\
\mathfrak{M}, s \Vdash_\eta \bot &\text{ if } s = \emptyset \\
\mathfrak{M}, s \Vdash_\eta \varphi \to \psi &\text{ if } \mathfrak{M}, t \Vdash_\eta \varphi \text{ implies } \mathfrak{M}, t \Vdash_\eta \psi \\
&\quad\ \text{for every } t \subseteq s \\
\mathfrak{M}, s \Vdash_\eta \varphi \wedge \psi &\text{ if } \mathfrak{M}, s \Vdash_\eta \varphi \text{ and } \mathfrak{M}, s \Vdash_\eta \psi \\
\mathfrak{M}, s \Vdash_\eta \varphi \vee\!\!\!\vee \psi &\text{ if } \mathfrak{M}, s \Vdash_\eta \varphi \text{ or } \mathfrak{M}, s \Vdash_\eta \psi \\
\mathfrak{M}, s \Vdash_\eta \forall x.\varphi &\text{ if } \mathfrak{M}, s \Vdash_{\eta[x \mapsto d]} \varphi \text{ for every } d \in \mathrm{D}_{\mathfrak{M}} \\
\mathfrak{M}, s \Vdash_\eta \exists\!\!\!\exists\, x.\varphi &\text{ if } \mathfrak{M}, s \Vdash_{\eta[x \mapsto d]} \varphi \text{ for some } d \in \mathrm{D}_{\mathfrak{M}}.
\end{aligned}$$

3 Natural Deduction

Before we discuss our target sequent calculi for (n-bounded) inquisitive logic, let us first have a look at a natural deduction system proposed by Ciardelli and Grilletti [3] (sound and complete for the same semantics) to see why it poses a challenge to a decidable notion of state. It includes rules for $\vee\!\!\!\vee$ and $\exists\!\!\!\exists$ from Figure 1, where α represents

$$\frac{\{(Y,\varphi)\,,\Gamma \Rightarrow \Delta, (Y,\psi) \mid Y \subseteq X\}}{\Gamma \Rightarrow \Delta, (X, \varphi \to \psi)}\ (\Rightarrow\to)$$

Figure 2: The rule $(\Rightarrow\to)$

a classical formula (i.e., one not involving these two constants). They are used many times in the completeness proof [3, Theorem 8.1]. Their soundness [4, Proposition 4.4.6] relies on the definition of the state

$$|\alpha|_{\mathfrak{M}} := \{w \in \mathrm{W}_{\mathfrak{M}} \mid \mathfrak{M}, w \Vdash_\eta \alpha\}$$

for a classical formula α and a model $\mathfrak{M}$. It is easy to show by a reduction from classical first-order logic that $\Vdash$ is undecidable. Therefore, we cannot use boolean predicates to represent states in order to formalize the soundness proof for this natural deduction system as we would not even be able to define $|\alpha|_{\mathfrak{M}}$.

4 Labelled Sequent Calculus

Litak and Sano [7, Table 2] provide a labelled sequent calculus and subsequently prove it to be sound [7, Proposition 4.8] and complete [7, Proposition 5.6] with respect to n-bounded semantics for each n (with a suitable cardinality restriction on labels)[2]. To illustrate the difference with the setup of Section 3, we discuss one clause from the Rocq mechanization [8] of the soundness proof: namely, the one corresponding to the rule presented in Figure 2.

Definition 5 (). We define the *set of labelled formulae* as $\mathcal{L} := \mathcal{P}_{\text{fin}}(\mathbb{N}) \times \mathcal{F}$.

A pair $\Gamma, \Delta \subseteq_{\text{fin}} \mathcal{L}$ is called a *sequent* and as usual denoted as $\Gamma \Rightarrow \Delta$.

Definition 6 (). Let $\mathfrak{M}$ be a model, $f\colon \mathbb{N} \to \mathrm{W}_{\mathfrak{M}}$, $\eta\colon \mathrm{Var} \to \mathrm{D}_{\mathfrak{M}}$, $X \subseteq_{\text{fin}} \mathbb{N}$ and $\varphi \in \mathcal{F}$. Then, we define the *support* for a labelled formula (X, φ) as follows:

$$\mathfrak{M}, f \Vdash_\eta (X, \varphi) \text{ if } \mathfrak{M}, f\,[X] \Vdash_\eta \varphi$$

Let $\mathfrak{M}$ be a model, $f\colon \mathbb{N} \to \mathrm{W}_{\mathfrak{M}}$, $\eta\colon \mathrm{Var} \to \mathrm{D}_{\mathfrak{M}}$ and $\Gamma, \Delta \subseteq_{\text{fin}} \mathcal{P}_{\text{fin}}(\mathbb{N}) \times \mathcal{F}$. We write $\mathfrak{M}, f \Vdash_\eta \Gamma \Rightarrow \Delta$ to denote that there is $(X, \varphi) \in \Delta$ s.t. $\mathfrak{M}, f \Vdash_\eta (X, \varphi)$ whenever $\mathfrak{M}, f \Vdash_\eta (X, \varphi)$ for all $(X, \varphi) \in \Gamma$.

Definition 7 (). Let $\Gamma, \Delta \subseteq_{\text{fin}} \mathcal{P}_{\text{fin}}(\mathbb{N}) \times \mathcal{F}$. We write $\Vdash \Gamma \Rightarrow \Delta$ if $\mathfrak{M}, f \Vdash_\eta \Gamma \Rightarrow \Delta$ for every $\mathfrak{M}$, $f\colon \mathbb{N} \to \mathrm{W}_{\mathfrak{M}}$ and $\eta\colon\ \mathrm{Var} \to \mathrm{W}_{\mathfrak{M}}$.

Now we can show that the right introduction rule for implication is indeed sound.

[2] In the paper, the signature is assumed to be purely relational. The Rocq formalization [8] provides an experimental extension with *rigid* function symbols (see also [5]).

Proposition 8 (). *Let $X \subseteq_{fin} \mathbb{N}$, $\varphi, \psi \in \mathcal{F}$, $\mathfrak{M}$ be a model, $f\colon \mathbb{N} \to \mathrm{W}_{\mathfrak{M}}$ and $\eta\colon \mathrm{Var} \to \mathrm{D}_{\mathfrak{M}}$ be a variable assignment. If $\mathfrak{M}, f \Vdash_\eta (Y, \varphi)\,, \Gamma \Rightarrow \Delta, (Y, \psi)$ for every $Y \subseteq X$, then we have $\mathfrak{M}, f \Vdash_\eta \Gamma \Rightarrow \Delta, (X, \varphi \to \psi)$.*

Proof. Without loss of generality, we assume that there is no other $(Z, \chi) \in \Delta$ such that $\mathfrak{M}, f \Vdash_\eta (Z, \chi)$. Consequently, we assume that $\mathfrak{M}, s \Vdash_\eta \varphi$ for some $s \subseteq f[X]$ and show $\mathfrak{M}, s \Vdash_\eta \psi$. By picking a label $Y \subseteq X$ such that $f[Y] = s$ we can conclude the proof in an obvious way. □

Rocq requires that an appropriate label Y is explicitly defined. In our approach, labels are implemented via lists in a suitable way. Therefore, Y must also be constructed as an explicit list. If the substate $s \subseteq f[X]$ does not come with a decision procedure, this is not possible.

5 Future Work

Once again, it appears that the availability of both `Prop`-based properties and `bool`-based predicates allows for careful choices tailored for specific formalisms, but may require forethought from the user. As inquisitive logic is a rapidly growing field still lacking formalizations in proof assistants (and generally tool support), we hope that our preliminary study paves the way for more work on the subject. The present formalization focuses mostly on complex semantic reasoning regarding the failure of *schematic validity* in the bounded case [7, Section 3] and experiments in derivability of sequents. Regarding the former, *op.cit.* notes that the very notion of a scheme used by Gabbay, Shehtman and Skvortsov [6, § 2.2–2.5] (following Bourbaki) appears to resemble (a second-order version of) *locally nameless* representation [1]. More work on mechanizing non-classical predicate logics may provide rewarding insights for all communities involved. As for the sequent calculus itself, it would be of interest (if more demanding) to fully mechanize metatheoretic results of Litak and Sano [7, Section 5–6], in particular syntactic cut elimination (Theorem 6.3). It is also worth mentioning that the ND calculus briefly discussed in Section 3 achieves completeness by means of heavily signature-dependent *cardinality formulae* [3, Section 6]. There seems to be no straightforward way of deriving them in the (n-bounded version of) sequent calculus of Litak and Sano [7], whereas the ND calculus of Ciardelli and Grilletti [3] does not seem tailored for deriving schematic validities; equipollence of both calculi is at present merely a consequence of their separate completeness theorems. Finally, it is an open question whether a sufficient condition for schematic validity given by Litak and Sano [7, Theorem 4.11], i.e., derivability without the atomic rule of the sequent calculus, is a necessary one. Settling such questions seems an interesting challenge for an extended version of our formalization and design choices we make herein, especially our treatment of signatures and arities (see Remark 2).

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