Turning the Coq Proof Assistant into a Pocket Calculator

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September 14, 2024

The Need for Guaranteed Mathematical Computations

Harald Helfgott on MathOverflow, 2013 (link)

I need to evaluate some (one-variable) integrals that neither SAGE nor Mathematica can do symbolically. As far as I can tell, I have two options:

a. Use GSL (via SAGE), Maxima or Mathematica to do numerical integration. This is really a non-option, since, if I understand correctly, the "error bound" they give is not really a guarantee.

b. Cobble together my own programs using the trapezoidal rule, Simpson's rule, etc., and get rigorous error bounds using bounds I have for the second (or fourth, or what have you) derivative of the function I am integrating. This is what I have been doing.

Is there a third option? Is there standard software that does (b) for me?

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The software suggested by the accepted answer computes an incorrect value on the example proposed by Helfgott.

Computing in Coq

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- Works poorly with abstract symbols:

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Compute (3 + 5)%R.
(* = R1 + (R1+R1) + (R1 + (R1+R1) * (R1+R1)): R *)
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```

Yet, if the result is known, one can do a formal proof

```
Goal (3 + 5 = 8)%R. Proof. ring. Qed.
```

Coq as a Pocket Calculator

Objectives

- Leverage the proof system of Coq.
- Give some meaningful answers to the user.
- Make it user-friendly.

Outline

Existential Variables and Tactic-in-Terms

$$3 + 5?$$

Leaving holes in the goal: existential variables

```
eassert (3 + 5 = ?[r]) as H.
{ (* 3 + 5 = ?r *)
  ring_simplify. reflexivity. }
(* H: 3 + 5 = 8 |- ... *)
```

Existential Variables and Tactic-in-Terms

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```

Leaving the whole goal as a hole: tactic-in-terms

```
Definition foo := ltac:(
  refine (_ : 3 + 5 = _) ;
  ring_simplify ; reflexivity).
Check foo. (* foo : 3 + 5 = 8 *)
```

A Rough Calculator

Tying things together

```
Ltac expand t := refine (_: (t = _));
  ring_simplify; reflexivity.

Definition foo := ltac:(expand (3 + 5)).
Check foo. (* 3 + 5 = 8 *)
```

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  ring_simplify; reflexivity.

Definition foo := ltac:(expand (3 + 5)).
Check foo. (* 3 + 5 = 8 *)
```

Unfriendly, but actually powerful

```
Definition bar x := ltac:(expand ((x+1) * (x-1))).
Check bar. (* forall x, (x+1) * (x-1) = x^2 - 1 *)
```

Interval Arithmetic in a Nutshell

Naive interval arithmetic

If $u \in [\underline{u}; \overline{u}]$ and $v \in [\underline{v}; \overline{v}]$,
then $u - v \in [\underline{u} - \overline{v}; \overline{u} - \underline{v}]$.

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Rigorous polynomial approximations: $f \in \langle P_f, \Delta_f \rangle_X$

If $u(x) - P_u(x) \in \Delta_u$ and $v(x) - P_v(x) \in \Delta_v$ for all $x \in X$,
then $(u - v)(x) - (P_u - P_v)(x) \in \Delta_u - \Delta_v$ for all $x \in X$.

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then $(u - v)(x) - (P_u - P_v)(x) \in \Delta_u - \Delta_v$ for all $x \in X$.

The CoqInterval library

```
Goal forall x, 0 <= x <= 1 -> sin (x + exp x) = 0 ->
     0.835 <= x <= 0.836.
Proof. intros x H E. root E. Qed.
```

Leveraging CoqInterval's Tactics

Interval arithmetic as a proof helper

```
interval_intro (PI^2/6) with (i_prec 10) as H.
(* H: 841/512 <= PI^2/6 <= 844/512 |- ... *)
```

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```

Leveraging the tactics

```
Definition foo := ltac:(
  let H := fresh in
  interval_intro (PI^2/6) with (i_prec 10) as H;
  exact H).
Check foo. (* 841/512 <= PI^2/6 <= 844/512 *)
```

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Making the tactics recognize goal evars

```
Definition foo :=
  ltac:(interval (PI^2/6) with (i_prec 10)).
```

Outline

1. Introduction
2. A rough calculator
3. Improving the user experience
   - Vernacular commands to the rescue
   - Postprocessing
4. Conclusion

Vernacular Commands to the Rescue

Objectives

1. Have a single short command per user query.
2. Make the proof term opaque.
3. Postprocess the type of the proof term.
4. (Improve performance.)

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Def and Do

```
Do interval (PI^2/6).
(* (PI ^ 2 / 6) ≃ 1.64493406685 *)
Def foo x ‘(0 <= x <= 1) := root (sin (x + exp x)).
(* x ≃ 0.835538085216 *)
```

Postprocessing the Proof Type

Why postprocess?

```
Def foo x ‘(0 <= x <= 1) := root (sin (x + exp x)).
(* x ≃ 0.835538085216 *)

Check foo.
(* forall x, 0 <= x <= 1 -> sin (x + exp x) = 0 ->
   7525858018462367 / 9007199254740992 <= x <=
   7525858018462401 / 9007199254740992 *)
```

Postprocessing the Proof Type

Why postprocess?

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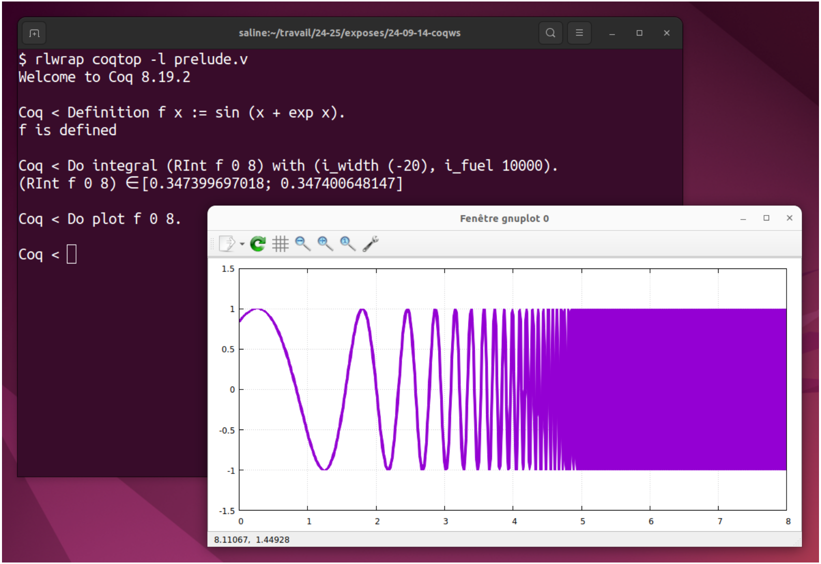
   7525858018462401 / 9007199254740992 *)
```

What about wide intervals?

```
Do integral (RInt (fun x => sin (x + exp x)) 0 8)
   with (i_width (-20), i_fuel 1000).
(* (RInt (fun x : R => sin (x + exp x)) 0 8)
   ∈ [0.347399697018; 0.347400648147] *)
```

Types are More Than Just Text



Outline

The Issue With Performance

Computations are performed thrice

```
Definition slow n := (fact n) mod (S n).

Ltac reduce v :=
  let w := eval vm_compute in v in
  exact_no_check (eq_refl w <: v = w).

Time Do reduce (slow 12). (* 102.7s *)

Time Eval vm_compute in slow 12. (* 34.2s *)
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Time Do reduce (slow 12). (* 102.7s *)

Time Eval vm_compute in slow 12. (* 34.2s *)
```

Naming expressions triggers memoization

```
Definition aux := slow 12.
Time Do reduce aux. (* 31.0s *)
```

Work in progress: Teach `Do`, `Def`, and the tactics how to create intermediate definitions.

Conclusion

The `Do` & `Def` commands

1. Invoke a given tactic to compute some proof term.
2. Postprocess and display the type of the proof term.

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"Is there standard software that does (b) for me?"

```
Do integral (RInt (fun x => Rabs (
        (x^4 + 10*x^3 + 19*x^2 - 6*x - 6) * exp x
      )) 0 1) with (i_relwidth 50).
(* (RInt (fun x => ...) 0 1) ≃ 11.1473105501 *)
```

Available in CoqInterval

https://coqinterval.gitlabpages.inria.fr/