An Encoding of (Co)inductive Types in Coq

via W- and M-types in the category of setoids

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The Coq Workshop 2023, July 31

Introduction

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Current state: an experiment

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Equality for coinductive types

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- impossible to prove `zeros = zeros'`
- need to define a bisimilarity relation by hand

```
CoInductive EqSt (s1 s2 : stream) : Prop := eqst {
  eqst_hd : hd s1 = hd s2;
  eqst_tl : EqSt (tl s1) (tl s2);
}.
```

Compositionnality

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- the automatically generated induction principle is useless
- impossible to abstract over `list`

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Context (F : Type -> Type)
Context (Fmap : ∀ (X Y : Type), (X -> Y) -> F X -> F Y).

Fail Inductive tree (A : Type) : Type :=
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- example: shuffle product on streams

```
Fail CoFixpoint shuffle (s s' : stream) : stream :=
  shuffle (tl s) s' + shuffle s (tl s').
```

Inspiration from other proof assistants

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- these frameworks require quotient types, propositional and functional extensionality
- goal: a similar framework, in Coq, axiom-free

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Functors of types

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Example:

$$F = \texttt{list} \qquad\qquad F^{\text{map}} = \texttt{List.map}$$

Constructors as algebras

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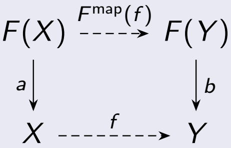
An *F-algebra* is a pair (X, a) where X is a type and $a : F(X) \to X$.

$$F(X) = \mathbf{1} + X \qquad\qquad \mathbf{1} + \mathrm{nat} \xrightarrow{[0,\mathrm{succ}]} \mathrm{nat}$$

Initial algebras

Definition (Initial F-algebra)

An *initial F-algebra* is an algebra $a : F(X) \to X$ such that for all algebra $b : F(Y) \to Y$, there exists a unique function $f : X \to Y$ such that the following diagram commutes:



Recursion via initiality

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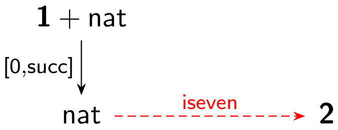

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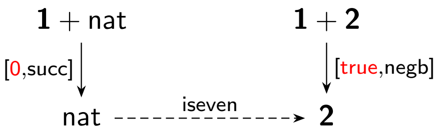

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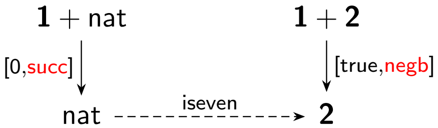

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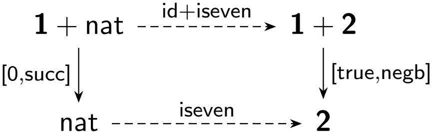

$$\begin{cases} \text{iseven}(0) = \text{true} \\ \text{iseven}(\text{succ}(n)) = \text{negb}(\text{iseven}(n)) \end{cases}$$

Destructors as coalgebras

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$$F(X) = A \times X \qquad \mathrm{stream}_A \xrightarrow{(\mathrm{hd},\mathrm{tl})} A \times \mathrm{stream}_A$$

Final coalgebras

Definition (Final F-coalgebra)

A *final F-coalgebra* is a coalgebra $c : Z \to F(Z)$ such that for all coalgebra $d : X \to F(X)$, there exists a unique function $f : X \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & Z \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle c} \\ F(X) & \overset{F(f)}{\dashrightarrow} & F(Z) \end{array}$$

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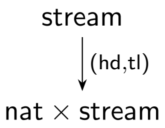

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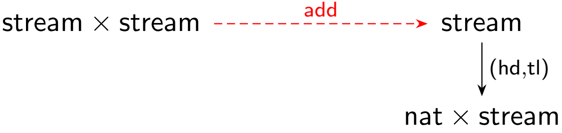

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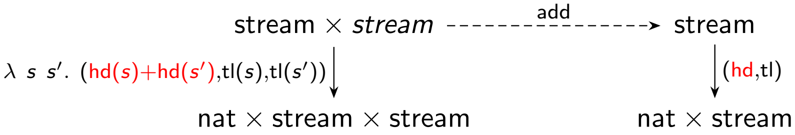

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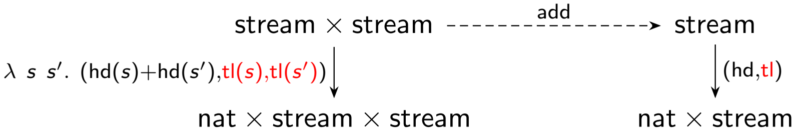

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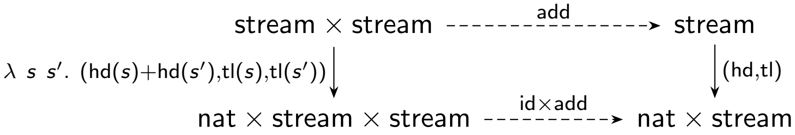

$$\begin{cases} \mathrm{hd}(\mathrm{add}(s, s')) = \mathrm{hd}(s) + \mathrm{hd}(s') \\ \mathrm{tl}(\mathrm{add}(s, s')) = \mathrm{add}(\mathrm{tl}(s), \mathrm{tl}(s')) \end{cases}$$

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- $\mathsf{colist}_A = \nu X.\ \mathbf{1} + A \times X$ potentially infinite lists
- $\mathsf{fbtree}_A = \nu X.\ A \times \mathsf{list}(X)$ finitely branching trees

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Polynomial functors

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$$P, Q ::= \mathrm{id} \mid \mathrm{cst}_S \mid P + Q \mid P \times Q \mid P^S$$

Definitions

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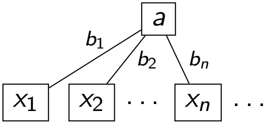

$a : A$

$f : B(a) \to X$

Examples

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$P(X) = \mathsf{option}(X) \simeq \mathbf{1} + X$	$\left(\mathbf{2} \triangleright \begin{array}{l} \mathsf{true} \mapsto \mathbf{0} \\ \mathsf{false} \mapsto \mathbf{1} \end{array}\right)$

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$P(X) = \mathsf{list}(X) \simeq \sum_{n:\mathsf{nat}} X^n$	$(\mathsf{nat} \triangleright n \mapsto \mathbf{n})$

Closure properties

Polynomial functors are closed under:

- sum
- product
- composition
- (μ and ν in the multivariate case)

which means providing the corresponding constructions on containers and establishing the associated functor equivalences

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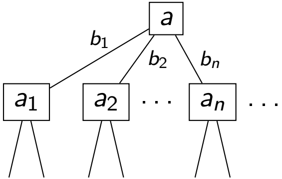

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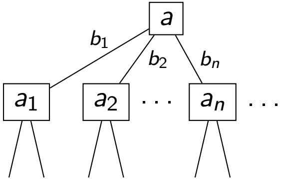

- well-founded trees: $W = \mu X.\ P(X)$

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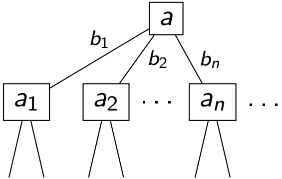

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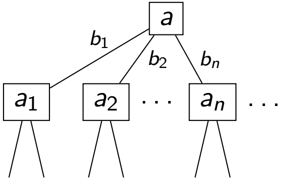

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"one (co)inductive to rule them all"

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- solution : setoids [Hofmann 1995]

Solution : shift of category

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0 1 2 $+$ $\times$ $\surd$

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$\boxed{\mathbf{0}\ \mathbf{1}\ \mathbf{2}\ +\ \times}\ \surd \qquad \boxed{A \to \text{Type}\ \sum\ \prod}\ ?$

Setoid families

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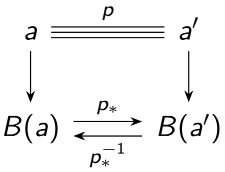
- *transport* function along an equivalence $p : a \equiv a'$, $p_* : B(a) \to B(a')$, isomorphism $B(a) \cong B(a')$



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Structure PFUNCTOR := {
  pf_func :> FUNCTOR SETOIDS SETOIDS;
  pf_cont : container;
  pfE : pf_func ≃ extension pf_cont }.
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W-setoids and M-setoids

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(implemented in the code)

In practice (1)

1. Define a `PFUNCTOR` using the provided DSL.

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2'. Use your own type, enriched with an equivalence relation, a (co)algebra structure and a proof that it is an initial/final (co)algebra of the functor.

In practice (2)

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5. A (co)induction principle is provided for proofs.

Future work

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- nested and mixed inductive-coinductive types $\to$ multivariate polynomial functors

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- quotients of polynomial functors
- more powerful (co)recursion principle $\rightarrow$ up-to techniques