Formalization of the Lebesgue Measure in MathComp-Analysis

Reynald Affeldt and Cyril Cohen

National Institute of Advanced Industrial Science and Technology (AIST), Tokyo, Japan

Université Côte d'Azur, Inria, Sophia Antipolis, France

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Motivation

Why formalize the Lebesgue measure?

1. Develop integration and probability theories on top of MathComp
2. More generally: development of reusable machinery for MathComp-Analysis [1, 2]
   - extension of MathComp for classical analysis (topology, real and complex analysis, etc.)

Problem statement

Construction of the Lebesgue measure which is:

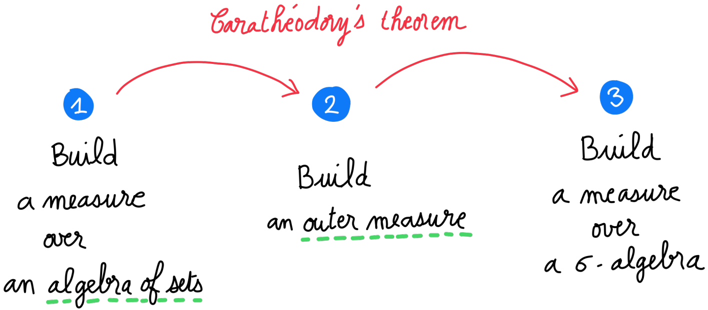
- a function μ
  - domain: sets that form a σ-*algebra*
  - codomain: extended real numbers (can be $+\infty$)
- which is a *measure*
  - in particular: it is σ-*additive*, i.e. $\mu(\bigcup_i F_i) = \sum_i \mu(F_i)$ when F_i are pairwise-disjoint

This is a long construction (our main reference [7] is 14 pages long and still glosses over many details)

This is non-trivial (the result can be admitted at the undergraduate-level in French universities)

Construction Approach

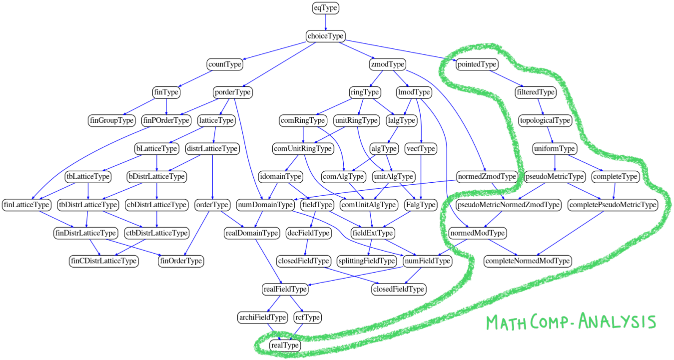
Standard, textbook approach, from the ground up:



Outline

Starting Point: MathComp and MathComp-Analysis

MathComp-Analysis adds several mathematical structures to MathComp:



Support for Extended Real Numbers

- $\infty - \infty$ is undefined in the mathematical practice
  - We define it as $-\infty$ so that the extended real numbers form a commutative monoid
  - **Important benefit:** we can use the bigop library of MathComp!
- Extended real numbers form a topological/uniform/pseudometric space
  - Tedious instantiation work using the following definition of ball:

```
Definition ereal_ball (x y : \bar R) (e : R) :=
  `|𝒞 x - 𝒞 y| < e.
```

  - **Reward:** we can reuse existing lemmas to develop the theory of sequences, e.g.:

```
Lemma ereal_limD (R : realType) (f g : (\bar R)^nat) :
  cvg f → cvg g → adde_def (lim f) (lim g) →
  lim (f \+ g) = lim f + lim g.
```

New Mathematical Structures

- *σ-algebra* (note the countable union *(**)*):

```
HB.factory Record isMeasurable T := {
 measurable : set (set T) ;
 measurable0 : measurable set0 ;
 measurableC : ∀ A, measurable A → measurable (~` A) ;
 measurable_bigcup : ∀ A : (set T)^nat,
  (∀ i, measurable (A i)) → measurable (\bigcup_i (A i)) (**) }.
```

- Algebra of sets (finite union only *(***)*):

```
HB.factory Record isAlgebraOfSets T := {
 measurable : set (set T) ;
 measurable0 : measurable set0 ;
 measurableC : ∀ A, measurable A → measurable (~` A) ;
 measurableU : ∀ A B, measurable A → measurable B →
  measurable (A `|` B) (***) }.
```

- These *factories* are implemented with Hierarchy-Builder [4]
  - *σ-algebras* actually extend algebras of sets, which extend *ring of sets*, which extend *semirings of sets*

Measures

- Measure:

```
Record axioms (mu : set T → \bar R) := Axioms {
  _ : mu set0 = 0 ;
  _ : ∀ x, measurable x → 0 ≤ mu x ;
  _ : semi_sigma_additive mu (**) }.
```

(**) $\stackrel{\text{def}}{=} \mu(\cup_n F_n) = \sum_i^\infty \mu(F_i)$ for any sequence F, s.t.

  - $\forall i, \texttt{measurable}(F_i)$
  - F_i pairwise-disjoint
  - $\texttt{measurable}(\cup_n F_n)$

- Outer measure:

```
Record axioms (mu : set T → \bar R) := Axioms {
  _ : mu set0 = 0 ;
  _ : ∀ x, 0 ≤ mu x ;
  _ : {homo mu : A B / A ⊆ B ↣ A ≤ B} ;
  _ : sigma_subadditive mu (***)}.
```

(***) $\stackrel{\text{def}}{=} \mu(\cup_n F_n) \leq \sum_i^\infty \mu(F_i)$ (no equality required when sets are pairwise-disjoint)

Outline

Carathéodory's Theorem (1/2)

- Goal: build a σ-*algebra* and a measure over it given an outer measure
- The resulting σ-*algebra* is composed of *Carathéodory measurable* sets, i.e., sets A s.t. $\forall X, \mu(X) = \mu(X \cap A) + \mu(X \cap \bar{A})$
- The resulting measure is over these Carathéodory measurable sets
- Coq proofs are a direct formalization of paper proofs
  - This is thanks to the newly developed support for sequences of extended real numbers!

Carathéodory's Theorem (2/2)

- Goal: define an outer measure μ^* given a measure μ over an *algebra of sets*

$$\mu^*(X) \stackrel{\text{def}}{=} \inf \left\{ \sum_i^{\infty} \mu(F_i) | (\forall i, \texttt{measurable}(F_i)) \wedge X \subseteq \bigcup_i F_i \right\}$$

- Again, Coq proofs are a direct formalization of paper proofs modulo a new development:
  - In the course of proving σ*-subadditivity*, we run into the following subgoal:

$$\mu^*(\cup_i F_i) \leq \sum_i^{\infty} \left(\mu^*(F_i) + \frac{\varepsilon}{2^i} \right)$$

  - The proof goes on by showing $\mu^*(\cup_i F_i) \leq \sum_{i,j} \mu(G_i j) \leq \sum_i \sum_j \mu(G_i\, j)$ for some well-chosen G, which requires *sums over general sets*

Sums (of non-negative terms) over General Sets

- Paper definition:

$$\sum_{i \in A} a_i \stackrel{\text{def}}{=} \sup \left\{ \sum_{i \in F} a_i \mid F \text{ non-empty finite subset of } A \right\}$$

- Coq definition:

```
Definition csum R (T : choiceType) (A : set T) (a : T → \bar R) :=
  if A == set0 then 0 else
  ereal_sup [set \sum_(i <- F) a i |
            F in [set F : {fset T} | [set i | i ∈ F] ⊆ A]].
```

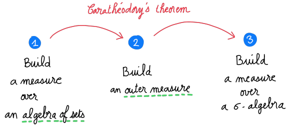
- Sample lemma:

$$(\forall n, 0 \leq a_n) \to (\forall k, J_k \neq \emptyset) \to J_k \text{ pairwise-disjoint} \to$$

$$\sum_{i \in \bigcup_{k \in K} J_k} a_i = \sum_{k \in K} \left(\sum_{j \in J_k} a_j \right)$$

Towards the Lebesgue Measure

- More formalized properties to complete Carathéodory's theorem:
  - The built measure coincides with the original measure
  - The built σ-*algebra* contains the smallest σ-*algebra* that contains the starting algebra of sets
  - The extension is unique provided the original measure is σ-finite

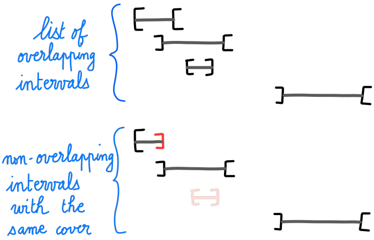


- Reminder:
  - Missing piece: a σ-finite measure over the algebra of sets generated by intervals...

Outline

The Algebra of Sets of Simple Sets

- Let us consider *simple sets*: sets that can be covered by a finite union of intervals
- Simple sets form an algebra of sets
    - Difficulty: stability by complement,
      better proved using non-overlapping intervals:



Decomposition of Intervals

Turn intervals into non-overlapping intervals with the same cover:

1. Ordering of intervals:

```
Definition lt_itv i j :=
  (i.1 < j.1)%O || ((i.1 == j.1) ∧ (i.2 < j.2)%O).
```

2. Recursive procedure to chop overlaps:

```
Lemma decompose_two i j t : decompose [:: i, j & t] =
  if (i ≤ j)%O then decompose (j :: t) else
  if (j ≤ i)%O then decompose (i :: t) else
  itv_diff i j :: decompose (j :: t).
```

3. Complete procedure:

```
Definition Decompose s :=
  decompose (sort le_itv [seq x ← s | neitv x]).
```

Length of Simple Sets

- Length of an interval:

```
Definition hlength (A : set R) : \bar R :=
 let i := Rhull A in i.2 - i.1.
```

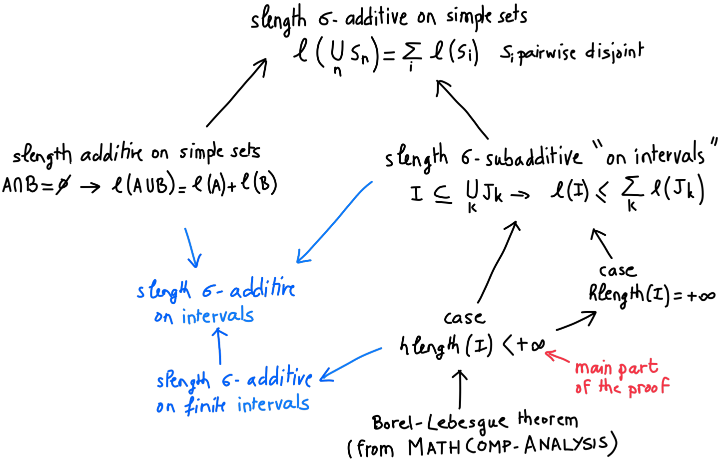
- Length of a simple set:

```
Definition slength (X : set R) : \bar R :=
 let s := xget [::] [set s | X = [sset of s] ] in
\sum_(i <- Decompose s) hlength (set_of_itv i).
```

- What is left to do?
  - prove that slength is σ-finite and that it is a measure
  - The difficulty is σ-*additivity*...

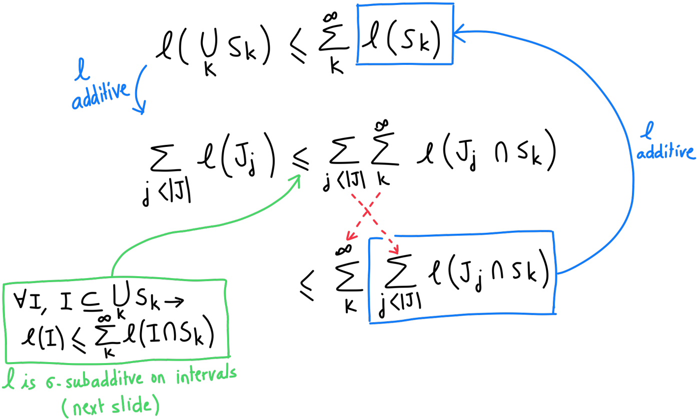
The σ-*additivity* of `slength`

Proof Overview



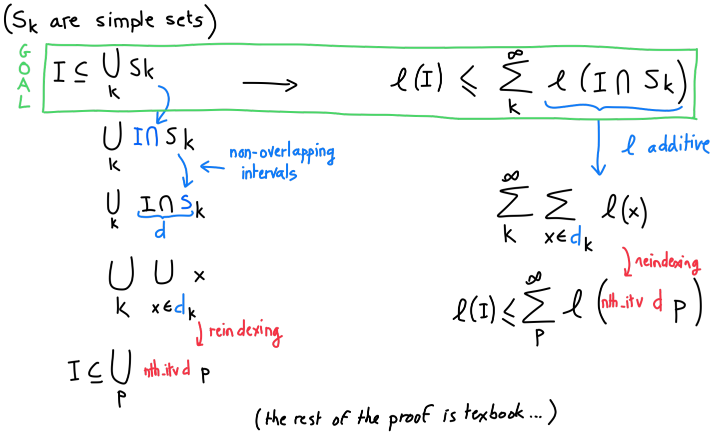
slength is σ-*additive*

Sample Technicality



slength is σ-*subadditive*

Sample Technicality



Outline

Related Work

About the Lebesgue measure:

- Lebesgue measure in Isabelle/HOL [6] defined using the gauge integral
- Lean defines the Lebesgue measure from the Lebesgue outer measure, not as an extension from an algebra of sets
- Lebesgue measure in Mizar as early as 1996 but needed a reconstruction in 2020 [5]

About Lebesgue integration in Coq:

- Recent work by Boldo et al. [3]: not directly compatible with our work (addition of extended real numbers is not associative because $\infty - \infty = 0$), Lebesgue measure is future work, σ-*algebra*'s defined as generated σ-*algebra*'s

Conclusions

We have a construction of the Lebesgue measure in MathComp-Analysis

- from the ground up (for documentation see, say, [7, 8, 9])
- principled construction
  - New mathematical structures (using Hierarchy-Builder)
  - Support for extended real numbers takes advantage of `bigop.v` and existing lemmas for topological structures
  - Sums of non-negative terms over general sets (using `finmap.v`)
  - Concrete instance of algebra of sets (using `interval.v`)

Current/future work: integration and probability theories

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