From guarded to well-founded

Formalizing Coq's guard condition

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Outline

1. Guard condition and trust
2. Translating guarded definitions
3. Current state of the implementation

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```
Fixpoint nat_rect (P : nat → Type) (f0 : P 0)
  (fS : forall n, P n → P (S n)) (n : nat) : P n :=
match n with
| O ⇒ f0
| S n' ⇒ fS n' (nat_rect P f0 fS n')
end.
```

Deep recursion

```
Inductive Even : nat → Prop :=
zeven : Even 0 | seven {n : nat} : Even n → Even (S (S n)).

Inductive Odd : nat → Prop :=
oneodd : (Odd (S 0)) | sodd {n : nat} : Odd n → Odd (S (S n)).

Definition EO (n : nat) := {Even n} + {Odd n}.

Definition aux {n : nat} (H : EO n) : EO (S (S n)) :=
match H with left p ⇒ left (seven p) | right p ⇒ right (sodd p) end.

Fixpoint evod (n : nat) : EO n :=
match n with
  | 0 ⇒ left zeven
  | S m ⇒
    match m with
      | 0 ⇒ right oneodd
      | S p ⇒ aux (evod p)
    end end.
```

The guard condition

We need to make sure that every `Fixpoint` terminates.

- Syntactic condition on the body of a `Fixpoint`.
- For each variable in the current context, track whether it is a subterm of the recursive argument.
- For each recursive call, check that the recursive argument is a subterm.

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- A `match` with certain restrictions can be a subterm if all branches are a subterm.
- . . . ?

$\Rightarrow$ We want a current justification of the guard condition.

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```
Definition nat_subterm : relation nat.
Theorem wf_nat_subterm : well_founded nat_subterm.

(* Large subterms. *)
Definition nat_subterm_eq : relation nat.

Definition nat_case (N : nat) (x : nat) (Hsub : nat_subterm_eq x N)
  (P : nat → Type) (f0 : P 0)
  (fS : forall (y : nat), nat_subterm y N → P (S y)) : P x.

Definition plus_body (n m : nat)
  (F : forall (n' m' : nat), nat_subterm n' n → nat) : nat.
```

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To check the guard on...

```
match c return p with
| C_i x y z ⇒ t
end
```

...check that `c` and `p` are guarded, then compute the recursive information on `x`, `y`, `z` depending on the information on `c`, and check `t` under this context.

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(fun x ⇒ t)
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...check that `t` is guarded under a context where `x` is not a subterm

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To check the guard on...

```
match c with
| C f ⇒ (fun x ⇒ t)
end y
```

...check that `t` is guarded under a context where `x` is not a subterm...unless there was some term applied to a `match` previously.

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- Recursively go through the function body, while collecting subterm information in the context.
- At a recursive call, check that we have a subterm in the recursive position.

To check the guard on...

```
f x y z
```

...when `f` is the function being defined, check that the argument in recursive position is a subterm.

The subterm relation

For each inductive type, we can systematically define a direct subterm relation.

```
Inductive nat : Set :=
O : nat | S : nat → nat.

Inductive nat_direct_subterm : relation nat :=
  nat_ds_1 : forall (n : nat), nat_direct_subterm n (S n).
```

[1] See for instance EQUATIONS

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The subterm relation is its transitive closure.

```
Definition nat_subterm : relation nat :=
  clos_trans nat nat_direct_subterm.
```

This is already done automatically by some tools[1], as well as proving its well-foundedness.

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Translation

```
Fixpoint evod (n : nat) : EO n :=
match n with
  | O ⇒ left zeven
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    match m with
      | O ⇒ right oneodd
      | S p ⇒ aux (evod p)
    end end.

Definition evod_body (n : nat)
  (F : forall (m : nat), nat_subterm m n → EO m) : EO n :=
  nat_case n n r_refl EO
    (left zeven)
    (fun m Hsub ⇒ nat_case n m (r_step Hsub) (fun m ⇒ EO (S m))
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In any case, there are some problems to solve with sorts.

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- The correct solution is not clear yet...
- For now, we inline the definition of the nested type.

```
Inductive Rose' : Set :=
| node' : listRose' → Rose'
with listRose' : Set :=
| nil' : listRose'
| cons' : Rose' → listRose' → listRose'.
```

- Ideally, we would like to reuse generically what was built for lists.

Evolution of the guard condition

```
Inductive True2 : Prop :=
I2: (False→ True2) → True2.

(* Using prop_ext : forall P Q, (P ↔ Q) → P = Q. *)
Theorem Heq: (False → True2) = True2.
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Fixpoint con (x : True2) : False :=
match x with
I2 f ⇒ con (match Heq in _=T return T with eq_refl ⇒ f end)
end.
```

Outline

Template Coq

- Quoting library for Coq.
- Allows the user to manipulate quotations and the global environment.
- Will include a certified checker for Coq.

Assuming completion of this work...

You only need to trust the well-foundedness of `Acc` to trust any recursive definition accepted by Template Coq.

Regular trees

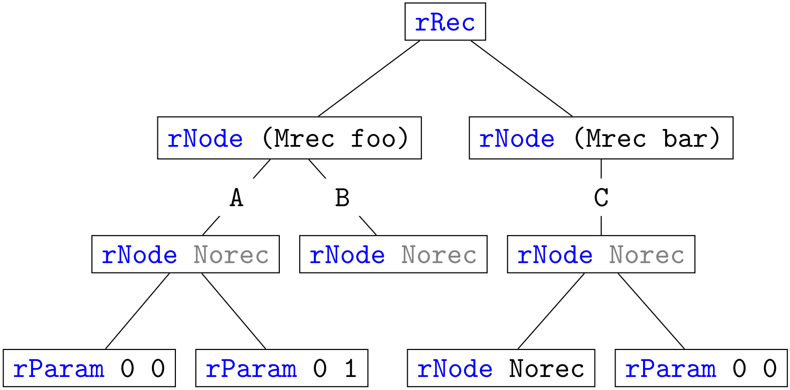
- Used to represent the recursive structure of a term.
- Infinite trees with finitely many subtrees.
- Represented in Coq's kernel by a finite datastructure with de Bruijn indices.

```
Inductive rtree (X : Set) : Set :=
| rNode (x : X) (sons : list (rtree X)) : rtree X
| rParam (i : nat) (j : nat) : rtree X
| rRec (j : nat) (defs : list (rtree X)) : rtree X.
```

From a client viewpoint, only see it as a tree of `rNode` with some payload of type `X`.

An example of regular tree

```
Inductive foo : Set :=
| A : foo → bar → foo
| B : foo
with bar : Set :=
| C : nat → foo → bar.
```



Unfolding of regular trees

A closed tree is either:

- some `rNode x sons` : the payload is `x` and its children are `sons`
- some `rRec j defs` : in that case we can consider the unfolding of this recursive node, that is `defs.(j)` where each `defs.(i)` is substituted for `rParam 0 i`.

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- Implement the translation of guarded functions.

Of course, prove all this correct all along.